Classroom Strategies to Make Sense and Persevere

A group of middle school teachers found that four strategies are effective when helping their students work through problems with understanding.

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Three middle-level mathematics teachers (grades 7 and 8) and a university mathematics educator formed a yearlong professional learning community (PLC) (DuFour, DuFour, and Eaker 2008). The objective was to collectively look at how we were promoting the Standards for Mathematical Practice (SMP) (CCSSI 2010) in our classes. Our monthly discussions followed an iterative cycle in which we continually shared instructional strategies and discussed their effectiveness in helping our students demonstrate the SMP (see **fig. 1**). We followed the PLC guidelines addressed in Hull, Miles, and Balka (2012) and identified various strategies to implement between our meetings, such as recording observations and writing reflections on our students' use of the practices. We then shared and analyzed our data at the next meeting.

As a team, we discussed which instructional strategies were effective and why they were effective in helping our students become more aware of their mathematical practice behaviors. In some cases, an instructional strategy needed to be enhanced or modified to promote more explicit use of the practices by students. For example, we decided to use the strategy of emphasizing perseverance themes and discussions in every class. In one meeting, we shared that simply discussing perseverance was not enough. Therefore, we decided to have our students keep a perseverance log (described later in the article). We found this iterative cycle helpful in guiding our discussions on enhancing the use of the math practices.

Although all the Standards for Mathematical Practice were included

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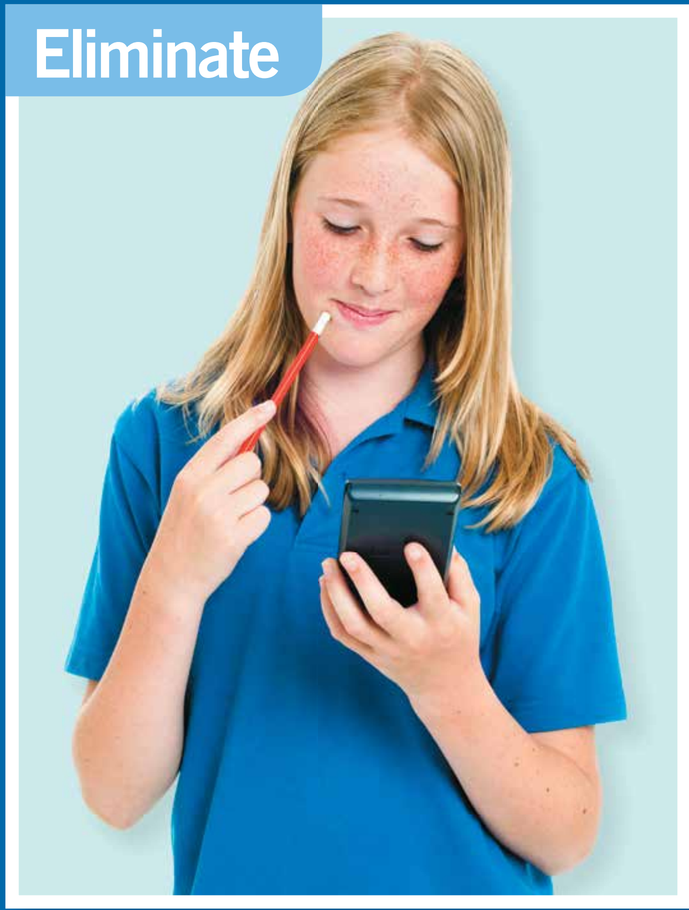
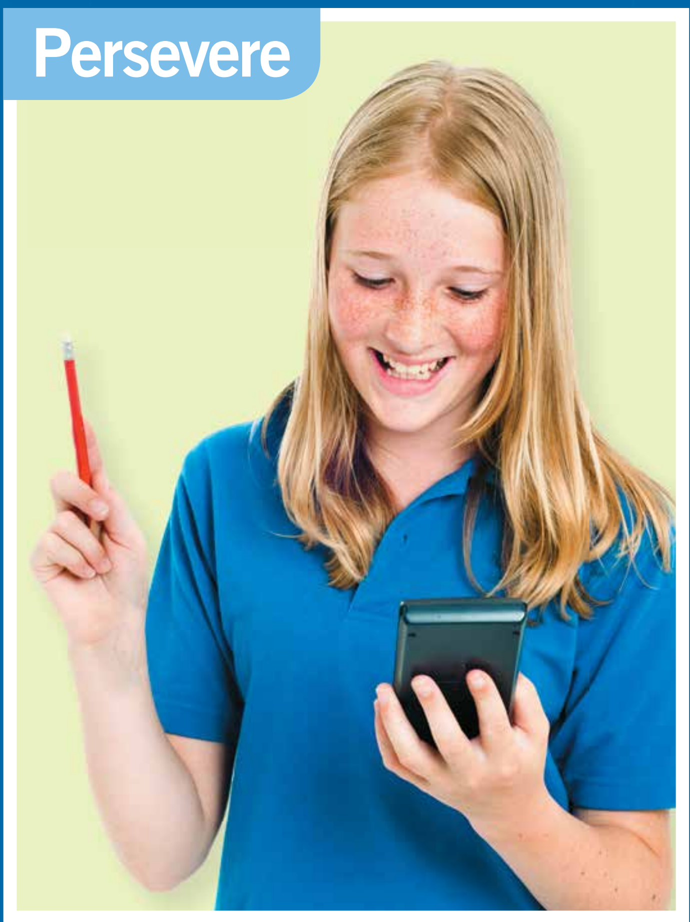




PAUL ABBITT/THINKSTOCK (2); FLASHGUN/THINKSTOCK (2)

Fig. 1 This iterative cycle illustrates strategies that promote student behaviors associated with the Standards for Mathematical Practice.

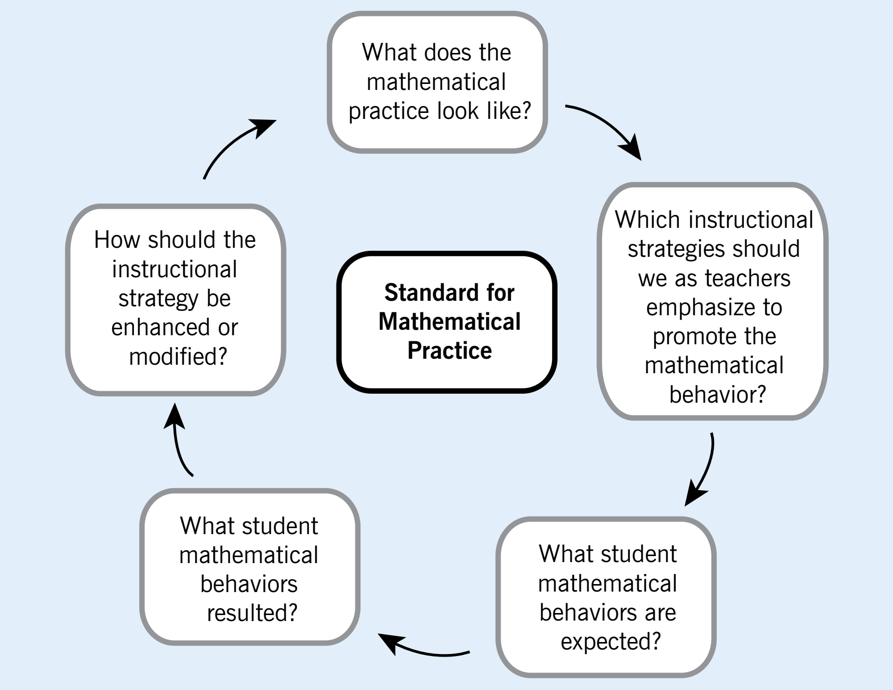


Fig. 2 Students discussed this problem and its solution.

Mr. Moreno went to the store to purchase some school clothes for his children during a "15% off" end-of-summer discount sale. After the discount, the price of one sweater is $11.50. What was the price of the sweater before the discount? Show all your work. Explain why you did each step.

Source: 4Sight Pennsylvania Benchmark (2007)

(a) Problem

$11.50 After discount
÷ .15
$76.66

My answer was $76.66 because 15% is basicly .15 so I divided that by 11.50 and got 76.66. After I got my answer I multiplied 76.66 by .15 and got 11.50.

(b) One student's solution

in our discussions, this article will share how the iterative cycle helped us identify four strategies that we found effective in our middle-level classrooms. In particular, these four strategies promoted SMP 1: "Make sense of problems and persevere in solving them" (CCSSI 2010, p. 6). We describe these strategies below with samples of student work.

STRATEGY 1: DOES IT MAKE SENSE?

"Mathematically proficient students check their answers using a different method, and they continually ask themselves, 'Does this make sense?' " (CCSSI 2010, p. 6).

How do we get students to reread a problem and check their answer to see if it makes sense? Although we often remind students to do just that, as well as check their answer using a different method, our students tended to rush through a problem just to say, "Done!" without revisiting the answer or the question posed. To show students the benefits of checking their answer to see if it makes sense, teacher A scanned student answers (without student names) to open-ended questions from the pre-algebra curriculum and put them into a PowerPoint® presentation. Class time was used by students to look at each answer and analyze why the solution did not make sense. Students discussed what a reasonable answer might be and the possible errors that may have been made that resulted in the incorrect answer. The teacher also asked the students to share possible strategies to find the solution. The following conversation occurred in the class when the teacher discussed problem A (see **fig. 2a**).

Teacher: Let's look at [the solution to] problem A [see **fig. 2b**]. Does this answer make sense? Take a minute and talk to your partners about the reasonableness of the solution.

[After approximately one minute, the teacher pulled the class together.]

Teacher: Group B, can someone share with the class whether you think the answer is reasonable?

Student: That answer is way too high. Why would he pay $76.66 for one sweater?

Teacher: Could a sweater actually cost $76.66?

Student: Yes, if it is a designer label or something.

Simply discussing perseverance was not enough; we asked our students to keep a Perseverance Log.

Teacher: So, why do you think $76.66 is too high?

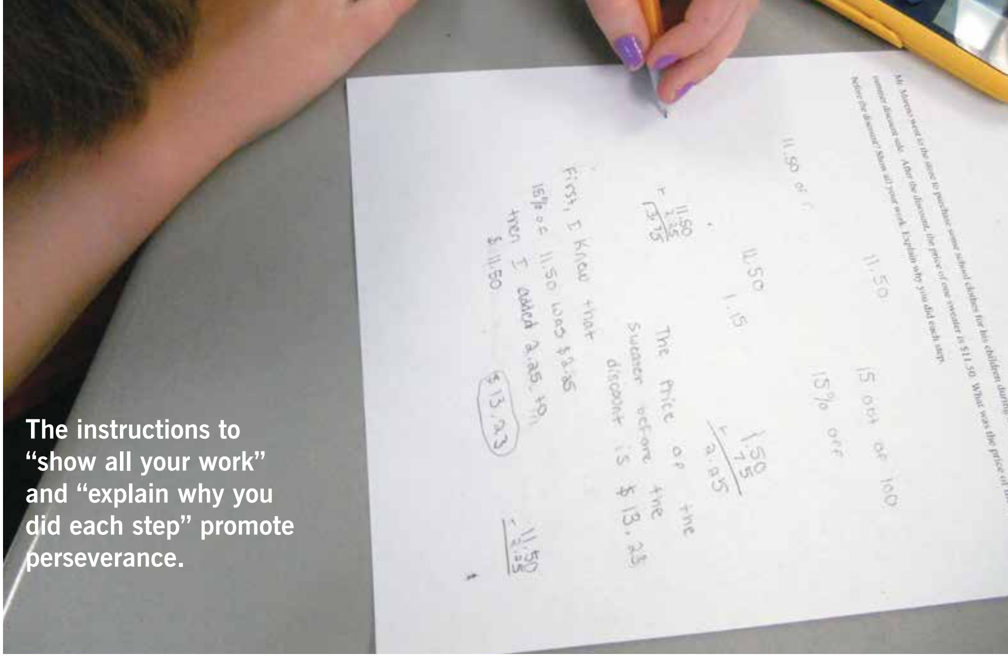
Student: Because the discounted price was only $11.50 and he saved 15 percent. If the original cost was $76.00, he would have saved over 50 percent.

Teacher: What could be a reasonable answer for this problem?

Student: Something a little higher than $11.50. Maybe between $13.00 and $18.00?

Teacher: How many of you believe this answer is unreasonable? [While looking around the room, she noticed that all the students agreed that the answer did not make sense.]

Teacher: What is another way we could use to check the answer?



The instructions to "show all your work" and "explain why you did each step" promote perseverance.

WILBURNE, WILDMANN, MORRET, AND STIPANOVIC

Student: You could start with the sweater being $76.66 and work backward, taking 15 percent off to see if you would get $11.50.

Teacher: Let's all try this and see if the answer checks.

The Does It Make Sense? strategy helped students focus strictly on the answer to a problem to determine whether it made sense. It also helped them share different ways of checking their answers. The more we introduced problems with incorrect answers, the more our students enjoyed the challenge of checking their reasonableness.

STRATEGY 2: THE PROCESS OF ELIMINATION (POE)

"Mathematically proficient students . . . make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt" (CCSSI 2010, p. 6).

The Process of Elimination (POE) strategy helped students think about the form and meaning of a solution on a multiple-choice problem, thus increasing the opportunity to solve the problem correctly. This strategy is helpful for students who are solving multiple-choice-type problems, such as those often found on state assessments. We wanted to use the POE strategy to help students think about and identify possible versus impossible solutions. The students were required to make a conjecture about the solution and plan a solution pathway to solve it. The following is a sample of the strategies used in one of our classrooms.

Teacher B presented this problem from prealgebra curriculum materials (4Sight Pennsylvania Benchmark 2007):

> At a manufacturing company, 80% of the employees work on the assembly line. If there are 720 assembly line workers, how many workers are there in the entire company?
>
> a. 576
> b. 600
> c. 786
> d. 900

After the students took a few minutes to think about the form and meaning of the solution, the teacher posed the following question.

Teacher: Which answers would not make sense for this problem?

Student A: (a) and (b)

Teacher: Tell us more about why choices (a) and (b) do not make sense as solutions.

Student A: Because there were already 720 people that work on the assembly line, so that means there

must be more than that in the entire company.

Teacher: Let's hear from another student. Which answers did you eliminate and why?

Student B: You can probably eliminate 900, too.

Teacher: Why can 900 be eliminated?

Student B: Well, if 720 is 80 percent of the employees, then there are only 20 percent more employees that work in the company. So, it looks like 786 would be a closer estimate.

Teacher: Does anyone else want to share their thoughts or respond to what student A or student B was thinking?

Student C: I know that 576 and 600 are definitely not the solution. But I would have to figure out the problem to see if there are 786 or 900 employees.

Teacher: OK, how would you solve the problem to determine whether there are 786 or 900 employees?

Student D: I would take 20 percent of 786 and see if I get 720. If that doesn't work, then I would take 20 percent of 900 to check the answer.

Teacher: Can someone respond to student D or offer another solution pathway?

Student E: If you take 20 percent of 786, you get 157.2. You need to multiply 786 by 0.80 and 900 by 0.80 to see which gives you 720.

Student A: Can you make an equation? Like, $720 = x(0.80)$?

Teacher: I see a few heads shaking. Student A, what are you thinking?

Student A: Yes, that works too.

We found that presenting multiple-choice problems with both reasonable and unreasonable solutions presented opportunities to have the class make conjectures about the form, meaning, and reasonableness of possible solutions. Also, by focusing on the answers and planning a solution pathway, the students were applying their estimating, mental math, and critical-thinking skills as emphasized in the Common Core State Standards for Mathematics.

Working collaboratively and checking the validity of solutions allows students to learn from one another.

WILBURNE, WILDMANN, MORRET, AND STIPANOVIC

STRATEGY 3: PERSEVERANCE LOGS

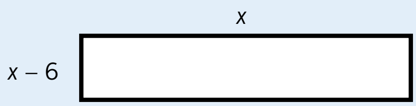
"Mathematically proficient students . . . analyze givens, constraints, relationships, and goals. . . . They monitor and evaluate their progress and change course if necessary" (CCSSI 2010, p. 6).

Our third strategy began by explicitly teaching students what it means to persevere and how to track and evaluate their progress. We created a Perseverance theme for our classrooms and shared examples of famous individuals and how and where perseverance paid off. For example, we explored Thomas Edison's quote, "I have not failed. I just have 10,000 ways that won't work" (www.goodreads.com), and discussed his attempts at inventing the lightbulb. We also touched on Milton Hershey's work to create a chocolate company. We wanted to emphasize to students that perseverance means sticking with it and not giving up. We also wanted to help them realize that concentrated effort and deep thinking are rewarding. Therefore, we focused on their successes when they exhibited the behavior of sticking with problems.

For this strategy, we selected open-ended problems that addressed the Common Core's standards for mathematical content. While they worked on these problems, we required our algebra 1 students to analyze the givens, constraints, relationships, and goals, as described in SMP 1. We asked students to keep a log describing how they persevered to solve problems and what they learned from the problems. We allowed them to work with a partner to discuss the givens and constraints, how they should be considered in the problem, and to share solution strategies. For example, students were presented with the problem in **figure 3**, which addressed the Common Core high school standard, "Solve linear equations and

Fig. 3 This problem addressed the CCSSM high school standard of solving linear equations.

Find all the values of x for which the perimeter of the figure below is at most 32 cm.



Source: Benson et al. (1995)

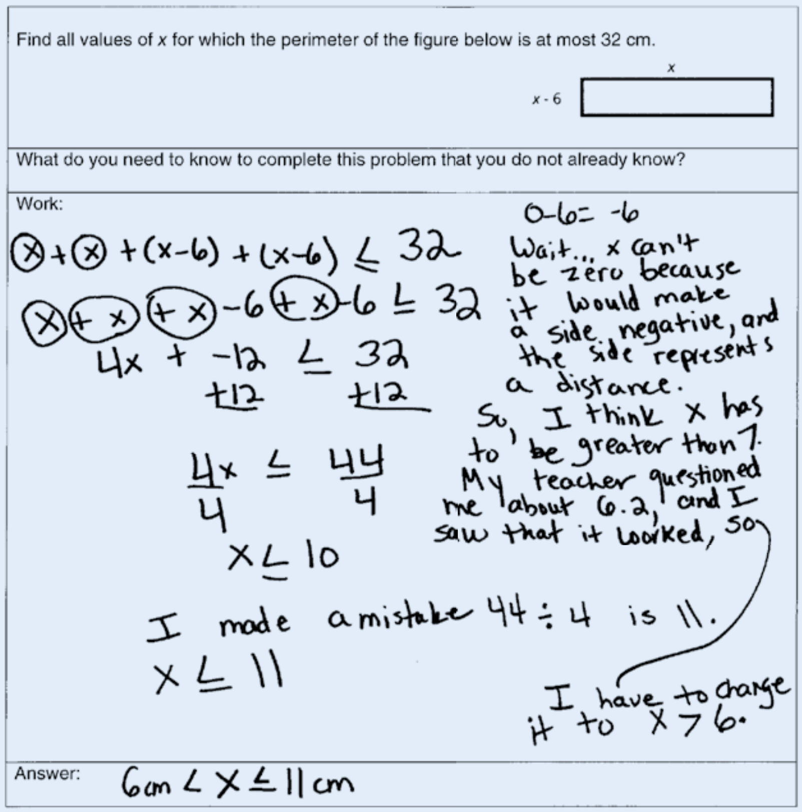
inequalities in one variable, including equations with coefficients represented by letters" (A-REI.3; CCSSI 2010, p. 65).

In one classroom, students were observed initially writing $2x - 6 = 32$ or $4x - 12 = 32$ to set up the problem. The teacher posed questions to pairs of students to help scaffold their thinking about the constraints for the values of x. She asked one pair, "What does 'at most' mean in the problem?" and "How would you represent 'at most' in mathematical form?" She wanted them to think about this question as she moved on to query another pair of students. The teacher continually reminded students to be sure they persevered and checked their solutions.

To review the problem, the teacher selected one pair of students to share their work at the board and describe how they worked through the problem. As they shared, several other students in the class could be overheard whispering to their partners, "We didn't even think about the fact that you can't have a negative side length" and "If we had checked all our answers, we would have found our mistake." Many students wrote in their logbooks that they thought they had persevered and found all the solutions but with further checking realized that the problem required only positive values for the lengths of the sides (see the examples in **fig. 4**).

We made a point to have students

Fig. 4 Keeping a Perseverance Log required that students analyze their solution processes and paths.



Find all values of x for which the perimeter of the figure below is at most 32 cm.

x

x - 6

What do you need to know to complete this problem that you do not already know?

Work:

$x + x + (x-6) + (x-6) \leq 32$

$x + x + x - 6 + x - 6 \leq 32$

$4x + -12 \leq 32$

$+12 \quad +12$

$\frac{4x}{4} \leq \frac{44}{4}$

$x \leq 10$

I made a mistake $44 \div 4$ is 11.

$x \leq 11$

$0 - 6 = -6$

Wait.. x can't be zero because it would make a side negative, and the side represents a distance.

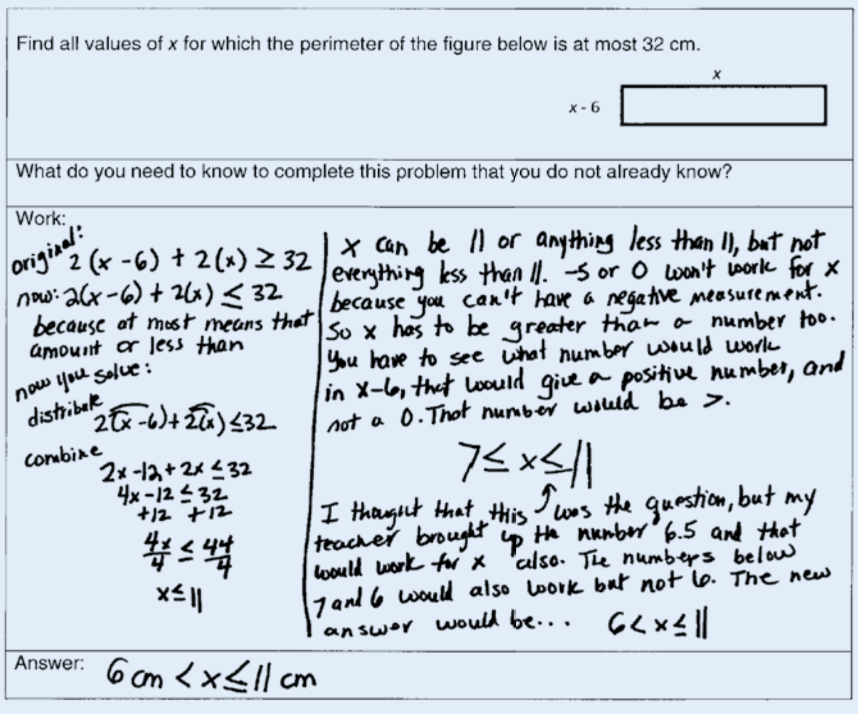
So, I think x has to be greater than 7.

My teacher questioned me about 6.2, and I saw that it worked, so

I have to change it to $x > 6$.

Answer: $6\text{cm} < x \leq 11\text{ cm}$

(a)



Find all values of x for which the perimeter of the figure below is at most 32 cm.

x

x - 6

What do you need to know to complete this problem that you do not already know?

Work:

original: $2(x-6) + 2(x) \geq 32$

now: $2(x-6) + 2(x) \leq 32$

because at most means that amount or less than

now you solve:

distribute $2(x-6) + 2(x) \leq 32$

combine $2x - 12 + 2x \leq 32$

$4x - 12 \leq 32$

$+12 \quad +12$

$\frac{4x}{4} \leq \frac{44}{4}$

$x \leq 11$

x can be 11 or anything less than 11, but not everything less than 11. −5 or 0 won't work for x because you can't have a negative measurement. So x has to be greater than a number too. You have to see what number would work in $x-6$, that would give a positive number, and not a 0. That number would be >.

$7 \leq x \leq 11$

I thought that this was the question, but my teacher brought up the number 6.5 and that would work for x also. The numbers below 7 and 6 would also work but not 6. The new answer would be... $6 < x \leq 11$

Answer: $6\text{ cm} < x \leq 11\text{ cm}$

(b)

WILBURNE, WILDMANN, MORRET, AND STIPANOVIC

Four Classroom Strategies to Encourage Sense Making and Persevering

1. Does It Make Sense?
Select student work that will require the class to think about why the solution is unreasonable. Present the problem and the solution to the class. Ask students to reflect on it individually, then to share their thoughts with a partner or small group about the reasonableness of the solution. Lead a class discussion about why the answer does not make sense. Make sure no student names are connected with the work that is posted and discussed.

2. The Process of Elimination (POE)
Use this strategy to help students identify possible versus impossible solutions. Provide four or five choices as possible solutions to a problem. Require students to think about the problem and determine which of the choices could be a reasonable solution. Encourage them to plan a solution pathway to solve the problem.

3. Perseverance Logs
Explore the theme of perseverance in every class. Discuss what perseverance means, and share examples of how various individuals became successful after persevering. Select examples from curriculum materials and other resources that require students to apply their conceptual understanding and knowledge of givens and constraints to solve the problems. Ask students to keep a Perseverance Log that includes the problems and how they persevered to solve them to help them recognize that struggling and perseverance are part of the learning process.

4. Analyzing Incorrect Responses
Highlight common errors that lead to unreasonable answers when solving problems. Hand out an index card or a sheet of paper containing a problem. Ask students to solve it and show their work. Collect the problems and quickly scan the solutions, sorting them into two stacks, correct and incorrect. Select one incorrect solution (excluding the student's name) and ask the class to discuss why the strategy or the solution does not make sense.

share how they persevered to find a solution in the mathematical discussions that followed each problem. Keeping a logbook of the different strategies they used to find a solution, how they checked their solution, and what they learned from each problem gave students a record of their efforts to persevere. Another benefit of keeping a logbook was that the students were able to identify connections between analogous problems.

STRATEGY 4: ANALYZING INCORRECT RESPONSES

"Mathematically proficient students . . . can understand the approaches of others to solving complex problems and identify correspondences between different approaches" (CCSSI 2010, p. 6).

This strategy is based on a video called "My Favorite No" (The Teaching Channel 2013), which became a favorite among the students. Our class periods began with a problem posted on the board or in a PowerPoint presentation. The students were given an index card or a sheet of paper and asked to take several minutes to solve the problem, showing all their work. The teacher collected the problems, quickly scanned the solutions, and sorted the student work into two stacks, correct and incorrect. The teacher selected either a single student's incorrect work or several students' incorrect works (without showing student names) to share with the class. Often, a student's work was selected because of the uniqueness of the incorrect strategy or because the student made a common mistake.

For example, to address the CCSSM Number System Standard 8.EE.1, "Know and apply the properties of integer exponents to generate equivalent numerical expressions" (CCSSI 2010, p. 54), prealgebra students were asked to simplify the following problem created by teacher B:

$$\frac{1}{6}(6 \div 3^{-2}) + 4 - |-8|$$

The teacher collected the students' work and quickly sorted the papers into two stacks, correct and incorrect. He selected one student's incorrect paper and posted it on the overhead projector (see **fig. 5**). The students were given a few minutes to look at the pathway used to solve the problem and were asked to find the error or misunderstanding. Several students shared their ideas and offered suggestions to correct the mistakes. In all our classes, we encouraged students to view "understanding the approaches of others" (SMP 1) and analyzing solution methods as learning experiences.

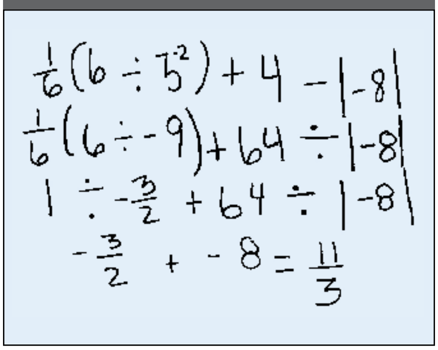
The Analyzing Incorrect Responses strategy helped our students recognize their errors in a discreet way and hear from classmates why an answer or a strategy did not make sense. The activity helped students think critically about the appropriateness of the mathematical process used and the reasonableness of the solution found. It also served as a great formative assessment to help monitor student misconceptions or difficulties with problems.

THE END RESULT: PRODUCTIVE LEARNING HABITS

We wanted our students to cultivate the behaviors associated with SMP 1 throughout our classrooms, so that these behaviors would become productive learning habits. By implementing these strategies, we helped our students explicitly focus on both the solutions and the strategies and think about whether they made sense. By the end of the academic year, we noticed fewer unreasonable answers and fewer students giving up on problems. Instead, students were circling their answers and often saying, "But does it make sense?" or "I found an answer, but it doesn't make sense to me." They were also recognizing both the need to persevere and how they persevered to solve a problem. The students often stated that they enjoyed problems that required them to persevere.

By implementing these four strategies (see the **sidebar** on p. 150) into our mathematics lessons, we demonstrated to students how SMP 1, "Make sense of problems and persevere in solving them" (CCSSI 2010, p. 6), can enhance their classwork and learning of mathematics. The use of these four strategies was a key factor in changing our middle school students' mathematical behavior and their thinking as they worked to find solutions to problems.

Fig. 5 This sample illustrates a solution that was under discussion as part of the Analyzing Incorrect Responses strategy.



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